A Generalization of the Space-Fractional Poisson Process and its Connection to some Lévy Processes

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Abstract

This paper introduces a generalization of the so-called space-fractional Poisson process by extending the difference operator acting on state space present in the associated difference-differential equations to a much more general form. It turns out that this generalization can be put in relation to a specific subordination of a homogeneous Poisson process by means of a subordinator for which it is possible to express the characterizing Lévy measure explicitly. Moreover, the law of this subordinator solves a one-sided first-order differential equation in which a particular convolution-type integral operator appears, called Prabhakar derivative. In the last section of the paper, a similar model is introduced in which the Prabhakar derivative also acts in time. In this case, too, the probability generating function of the corresponding process and the probability distribution are determined.

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1 Introduction and background

In the last decade, it became apparent that several phenomena that can be modeled in terms of point processes are non Poissonian in nature (see [2, 21] as examples). As a consequence, there has been an increased interest in generalizing the Poisson process $N(t)$. The Poisson process is a counting process with many nice properties. It is a Lévy process and, therefore, its increments are time-homogeneous and independent. It is a renewal process, meaning that the sojourn times between points (or events) are independent and identically distributed following the exponential distribution. It is a birth-death Markov process and its counting probability obeys the forward Kolmogorov equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t), \tag{1.1}$$

where $p_k(t) = \mathbb{P}\{N(t) = k\}$, $k \geq 0$, $t \geq 0$ are the state probabilities of the Poisson process and λ is the rate of the Poisson process. There are many possible ways to generalize this process. We are interested in the so-called fractional generalizations of the Poisson process where either the derivative on the left-hand side or the difference equation on the right-hand side of (1.1) are replaced by suitable fractional operators.

For instance, if one keeps the renewal property and considers sojourn times such that $\mathbb{P}\{N_\beta(t) = 0\} = E_\beta(-t^\beta)$ for $0 < \beta < 1$, where $E_\beta(z) = \sum_{k=0}^{\infty} z^k/\Gamma(k\beta + 1)$ is the one-parameter Mittag-Leffler function, one gets the renewal fractional Poisson process $N_\beta(t)$ discussed in [30]. This leads to the equation for $p_k(t) = \mathbb{P}\{N_\beta(t) = k\}$ [28],

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t), \qquad k \geq 0,\ t \geq 0, \tag{1.2}$$

(with $\lambda = 1$) where $\mathrm{d}^\beta/\mathrm{d}t^\beta$ is the so-called Caputo derivative, a pseudo-differential operator defined as

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-u)^{-\beta} \frac{\mathrm{d}f(u)}{\mathrm{d}u}\, \mathrm{d}u. \tag{1.3}$$

The renewal fractional Poisson process is not a Lévy process [36].

Another possibility is generalizing the Poisson process via the so-called space-fractional Poisson process studied in [33]. The space-fractional Poisson process is in practice a homogeneous Poisson process subordinated to an independent stable subordinator. We describe this model in details in Section 3.1. Here we aim at a further generalization of this process preserving the Lévy property. This generalized process is constructed by means of a superposition of suitably weighted independent space-fractional Poisson processes. The resulting process is then subordinated by means of a random time process in order to account for the modelization of a possible irregular flow of time. The complete construction of the model is done in Section 3.2.

The motivation at the basis of this study lies in the importance that superposition of, possibly dependent, point processes plays in applied sciences. An important example concerns the modelling of neurons' incoming signals. It is commonly accepted that each neuron obtains information from the neighbouring neurons in the form of *spike trains* (i.e. signals with powerful bursts), closely resembling realizations of stochastic point processes. Therefore, each neuron, receives a superposition of, possibly rescaled, point processes and its subsequent behaviour depends on the characteristics of this afferent combined input signal. Based on classical result contained in [11, 19, 16, 9] (roughly saying that a superposition of sufficiently sparse independent point processes converges to a Poisson process), the input signal was considered in this applied literature to be well approximated by a Poisson process [see e.g. 20, 38]. However, it was later shown that the above result does not always apply to superposition of signals from neurons' activity and that experimental evidence deviates from a Poissonian structure [29, 8, 12]. Recently, the study of weak convergence of superposition of point processes has regained interest [see e.g. 10, and the references therein] showing that different behaviours are possible. Within this framework, we can consider the model we are going to describe as a weighted finite superposition of independent space-fractional Poisson processes (each of them generalizing the homogeneous Poisson process but also admitting the possibility of jumps of any integer order). Each of these space-fractional Poisson processes can be thought to model different groups of neurons (different areas of the brain) acting together with simultaneous spikes giving rise to the non-unitary jumps. Notice, finally, that the space-fractional Poisson process is a non-renewal process and so is the generalized space-fractional Poisson process. This is a key feature for the combined neurons' input signal as it is remarked in [29].

For the sake of clarity and simplicity, we first recall some selected basic mathematical results regarding subordinators which will be useful in the following. Section 2 presents the construction of a random time-change by means of independent subordinators and tempered subordinators. This is of fundamental importance for the definition of the generalized space-fractional Poisson process which will be carried out in Section 3.

We start thus by recalling some basic facts on subordinators and tempered stable subordinators. The reader can refer to [5] or [26] for a more in-depth explanation. We recall that a subordinator is an increasing Lévy process defined as follows.

Definition 1.1. *Given a filtered probability space $(\Omega, \mathscr{F}, \mathfrak{F}, \mathbb{P})$, where $\mathfrak{F} = (\mathscr{F}_t)_{t\geq 0}$ is the associated right-continuous filtration, the process S_t, $t \geq 0$, adapted to $\mathfrak{F}$, starting from zero, and with increasing paths, is called a subordinator if it has independent and stationary increments or, equivalently, $\forall s > t \geq 0$, $S_t - S_s$ is independent of $\mathscr{F}_s$ and $S_t - S_s = S_{t-s}$ in distribution.*

In this paper we consider only strict subordinators that is those with infinite lifetime and with the only cemetery state (∞) located at $t = \infty$. For this special class of subordinators a simplified version of the well-known Lévy–Khintchine formula for general Lévy processes holds. In particular, the following theorem gives a characterization of subordinators in terms of Laplace transforms of their one dimensional law. Let us first recall that, thanks to stationarity and independence of the increments, we have that

$$\mathbb{E} \exp(-\mu S_t) = \exp\left[-t\Phi(\mu)\right], \qquad t \geq 0,\ \mu \geq 0, \tag{1.4}$$

where $\Phi(\mu)$ is called the Laplace exponent of the process S_t, $t \geq 0$. Different expressions for $\Phi(\mu)$ give rise to different subordinators, but not any functional form is allowed. This is exactly what the Lévy–Khintchine formula for subordinators tells us.

Theorem 1.1. *Any function $\Phi(\mu)$, $\mu \geq 0$, that can be put in the unique form*

$$\Phi(\mu) = b\mu + \int_0^\infty \left(1 - e^{-\mu x}\right) m(\mathrm{d}x), \qquad b \geq 0, \tag{1.5}$$

where $m(\mathrm{d}x)$ is a measure concentrated on $(0, \infty)$ such that $\int_0^\infty (1 \wedge x) m(\mathrm{d}x) < \infty$, is the Laplace exponent of a strict subordinator. Conversely, if $\Phi(\mu)$ is the Laplace exponent of a strict subordinator, there exist a non negative number b and a unique measure m with $\int_0^\infty (1 \wedge x) m(\mathrm{d}x) < \infty$ such that (1.5) holds true.

For a detailed proof see [5]. The measure m is called the Lévy measure and b is the drift associated to the strict subordinator S_t, $t \geq 0$. For the sake of simplicity, in the following we shall use the term subordinator meaning in fact a strict subordinator.

A particularly simple example of a subordinator is the so-called stable subordinator. This is characterized by the Laplace exponent $\Phi(\mu) = \mu^\alpha$, $\alpha \in (0,1)$, corresponding to a Lévy measure $m(\mathrm{d}x) = [\alpha/\Gamma(1-\alpha)]\, x^{-1-\alpha}\mathrm{d}x$. Note that the case $\alpha = 1$ is omitted as it is trivial. The one-parameter α-stable subordinator is at the basis of the probabilistic theory of anomalous diffusion based on subordination. In fact, it is a building block for the numerous processes connected to fractional evolution equations and their generalizations [32, 15] and also for other processes connected for example to point processes [33, 4]. Moreover, the importance of stable subordinators stems also from the fact that they are scaling limits of some totally skewed generalized random walks [31]. Given a subordinator S_t, $t \geq 0$, it is possible to define its right-inverse process [6] as

$$E_t = \inf\{w > 0 \colon S_w > t\}, \qquad t \geq 0. \tag{1.6}$$

The process E_t is non-Markovian with non-stationary and dependent increments [40, 27].

Let us now introduce some basic facts on tempered subordinators, in particular for tempered stable-subordinators. First of all, let us refer the reader to the paper [37] for a complete and detailed account on the general theory of tempered stable processes. However, it is interesting to note that tempered models already appeared in the literature (amongst others, for example, the KoBoL model [25, 7]). The class of tempered stable subordinators has been introduced in order to have processes possessing nicer properties than those of standard stable subordinators. In practice the Lévy measure of a stable subordinator is exponentially tempered, thus obtaining

$$m(\mathrm{d}x) = e^{-\xi x}\frac{\alpha}{\Gamma(1-\alpha)}x^{-1-\alpha}\mathrm{d}x, \qquad \alpha \in (0,1),\ \xi > 0. \tag{1.7}$$

This simple operation produces desirable results as seen immediately by realizing that the Laplace exponent in this case can be written as $\Phi(\mu) = (\xi+\mu)^\alpha - \xi^\alpha$. For more information regarding tempered stable subordinators and associated differential equations, see [1]. In Section 2, we introduce a process which has both the standard stable subordinator and the tempered stable subordinator as special cases and we study its properties. In Section 3, instead, we describe connections of the introduced process with some difference-differential equations involving a generalized fractional difference operator acting in space. An interesting special case related to these equations is that regarding the so-called space-fractional Poisson process [33]. In the last Section 3.3, we present a further generalization leading to a process which is no longer a Lévy process. This generalization is based on the study of similar difference-differential equations but involving the so-called regularized Prabhakar derivative in time that generalizes the Caputo derivative in time.

2 A subordinated sum of independent stable subordinators

Let us consider the filtered probability space $(\Omega, \mathscr{F}, \mathfrak{F}, \mathbb{P})$ and the process

$${}_\eta\mathcal{V}_t^{\nu,n} = \sum_{r=1}^n \binom{n}{r}^{\frac{1}{\nu r}} \eta^{\frac{n-r}{\nu r}} V_t^{\nu r} = \sum_{r=1}^n V^{\nu r}_{\binom{n}{r}\eta^{n-r}t}, \quad t \geq 0,\ n \in \mathbb{N},\ \eta > 0,\ \nu r \in (0,1),\ r = 1,\dots,n, \tag{2.1}$$

where $V_t^{\nu r}$, $t \geq 0$, $r = 1,\dots,n$, are n $\mathfrak{F}$-adapted independent νr-stable subordinators. Let us further consider a positive real parameter δ such that $\lceil \delta \rceil = n$ and the $\mathfrak{F}$-adapted tempered δ/n-stable subordinator $\mathcal{V}_t^{\delta/n}$, $t \geq 0$, evaluated at η^n, i.e. with Laplace exponent

$$\Phi(\mu) = \left(\eta^n + \mu\right)^{\delta/n} - \eta^\delta, \qquad \delta > 0,\ \eta > 0. \tag{2.2}$$

Proposition 2.1. *The Laplace transform of ${}_\eta\mathcal{V}_t^{\nu,n}$, $t \geq 0$, reads*

$$\mathbb{E}\exp\left(-\mu\, {}_\eta\mathcal{V}_t^{\nu,n}\right) = \exp\left(-t\left[(\eta+\mu^\nu)^n - \eta^n\right]\right), \qquad t \geq 0,\ \mu > 0. \tag{2.3}$$

Proof. Formula (2.3) can be proven by noticing that

$$\mathbb{E}\exp\left(-\mu\, {}_\eta\mathcal{V}_t^{\nu,n}\right) = \prod_{r=1}^n \mathbb{E}\exp\left(-\mu\binom{n}{r}^{\frac{1}{\nu r}}\eta^{\frac{n-r}{\nu r}}V_t^{\nu r}\right) \tag{2.4}$$

and by recalling that $\mathbb{E}\exp\left(-\mu V_t^\beta\right) = \exp\left(-t\mu^\beta\right)$. We thus obtain

$$\mathbb{E}\exp\left(-\mu\,{}_\eta\mathcal{V}_t^{\nu,n}\right) = \prod_{r=1}^{n}\exp\left(-t\binom{n}{r}\eta^{n-r}\mu^{\nu r}\right) \tag{2.5}$$

and hence the claimed result. □

Let us now consider the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and the subordinated filtration $\mathfrak{G} = (\mathscr{G}_t)_{t\geq 0} = (\mathscr{F}_{\mathcal{V}_t^{\delta/n}})_{t\geq 0}$. In the following, our interest will focus on the subordinated and $\mathfrak{G}$-adapted process

$${}_\eta\mathfrak{V}_t^{\nu,\delta} = {}_\eta\mathcal{V}^{\nu,n}_{\mathcal{V}_t^{\delta/n}}, \qquad t \geq 0,\ \delta > 0,\ \nu \in (0,1),\ \eta > 0. \tag{2.6}$$

Proposition 2.2. *The Laplace transform of the process ${}_\eta\mathfrak{V}_t^{\nu,\delta}$, $t \geq 0$, can be written as*

$$\mathbb{E}\exp\left(-\mu\,{}_\eta\mathfrak{V}_t^{\nu,\delta}\right) = \exp\left(-t\left[\left(\eta+\mu^\nu\right)^\delta - \eta^\delta\right]\right), \qquad t \geq 0,\ \mu > 0. \tag{2.7}$$

Proof. By definition (2.6) we can write that

$$\begin{aligned}\mathbb{E}\exp\left(-\mu\,{}_\eta\mathfrak{V}_t^{\nu,\delta}\right) &= \int_0^\infty \mathbb{E}\exp\left(-\mu\,{}_\eta\mathcal{V}_s^{\nu,n}\right)\mathbb{P}\left(\mathcal{V}_t^{\delta/n} \in \mathrm{d}s\right) \\ &= \int_0^\infty \exp\left(-s\left[\left(\eta+\mu^\nu\right)^n - \eta^n\right]\right)\mathbb{P}\left(\mathcal{V}_t^{\delta/n} \in \mathrm{d}s\right).\end{aligned} \tag{2.8}$$

Then by using (2.2) we directly arrive at (2.7). □

Remark 2.1. *Plainly, when $\delta = n = 1$, the process* (2.6) *coincides with a standard ν-stable subordinator, while for $\nu \to 1$, $\delta \in (0,1)$, it is a tempered δ-stable subordinator. As an example, we also note that for $\delta = 2$, $\nu \in (0,1/2)$, ${}_\eta\mathfrak{V}_t^{\nu,\delta} = V_t^{2\nu} + V_{2\eta t}^{\nu}$ is the rescaled sum of two independent stable subordinators.*

In the following theorem we prove that the $\mathfrak{G}$-adapted process ${}_\eta\mathfrak{V}_t^{\nu,\delta}$, $t \geq 0$, is in fact a subordinator and derive its associated Lévy measure.

Theorem 2.1. *The $\mathfrak{G}$-adapted process ${}_\eta\mathfrak{V}_t^{\nu,\delta}$, $t \geq 0$, is a subordinator. Furthermore, its associated Lévy measure is*

$$m(\mathrm{d}x) = \int_0^\infty e^{-\eta^n t}\,\mathbb{P}\left({}_\eta\mathcal{V}_t^{\nu,n} \in \mathrm{d}x\right)\frac{\delta}{n}\frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\frac{\delta}{n}\right)}\mathrm{d}t. \tag{2.9}$$

Proof. In order to prove the statement of the theorem we make use of the Lévy–Khintchine formula for subordinators. The Laplace exponent associated to ${}_\eta\mathfrak{V}_t^{\nu,\delta}$, $t \geq 0$, that is $\Phi(\mu) = \left(\eta+\mu^\nu\right)^\delta - \eta^\delta$, can be retrieved with the following steps.

$$\begin{aligned}&\int_0^\infty \left(1-e^{-\mu x}\right)\int_0^\infty e^{-\eta^n t}\,\mathbb{P}\left({}_\eta\mathcal{V}_t^{\nu,n} \in \mathrm{d}x\right)\frac{\delta}{n}\frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\frac{\delta}{n}\right)}\mathrm{d}t \\ &= \int_0^\infty \mathrm{d}t\, e^{-\eta^n t}\frac{\delta}{n}\frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\frac{\delta}{n}\right)}\int_0^\infty \left(1-e^{-\mu x}\right)\mathbb{P}\left({}_\eta\mathcal{V}_t^{\nu,n} \in \mathrm{d}x\right) \\ &= \int_0^\infty \mathrm{d}t\, e^{-\eta^n t}\frac{\delta}{n}\frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\frac{\delta}{n}\right)}\left[1-\int_0^\infty e^{-\mu x}\,\mathbb{P}\left({}_\eta\mathcal{V}_t^{\nu,n} \in \mathrm{d}x\right)\right] \\ &= \int_0^\infty \mathrm{d}t\, e^{-\eta^n t}\frac{\delta}{n}\frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\frac{\delta}{n}\right)}\left(1-e^{-t\left[\left(\eta+\mu^\nu\right)^n-\eta^n\right]}\right) \\ &= \int_0^\infty \mathrm{d}t\left(e^{-\eta^n t} - e^{-t\left(\eta+\mu^\nu\right)}\right)\frac{\delta}{n}\frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\frac{\delta}{n}\right)}\end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&= \int_0^\infty \mathrm{d}t \left(1-e^{-t(\eta+\mu^\nu)}\right) \frac{\delta}{n} \frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\frac{\delta}{n}\right)} - \int_0^\infty \mathrm{d}t \left(1-e^{-\eta^n t}\right) \frac{\delta}{n} \frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\frac{\delta}{n}\right)} \\
&= \frac{\delta}{n} \frac{1}{\Gamma\left(1-\frac{\delta}{n}\right)} \left[\int_0^\infty \int_0^t \left(\eta+\mu^\nu\right)^n e^{-y\left(\eta+\mu^\nu\right)^n} \mathrm{d}y \, t^{-1-\frac{\delta}{n}} \mathrm{d}t - \int_0^\infty \int_0^t \eta^n e^{-y\eta^n} \mathrm{d}y \, t^{-1-\frac{\delta}{n}} \mathrm{d}t \right] \\
&= \frac{\delta}{n} \frac{1}{\Gamma\left(1-\frac{\delta}{n}\right)} \left[\int_0^\infty \int_y^\infty \left(\eta+\mu^\nu\right)^n e^{-y\left(\eta+\mu^\nu\right)^n} \mathrm{d}y \, t^{-1-\frac{\delta}{n}} \mathrm{d}t - \int_0^\infty \int_y^\infty \eta^n e^{-y\eta^n} \mathrm{d}y \, t^{-1-\frac{\delta}{n}} \mathrm{d}t \right] \\
&= \frac{\delta}{n} \frac{1}{\Gamma\left(1-\frac{\delta}{n}\right)} \int_0^\infty \mathrm{d}y \left(\eta+\mu^\nu\right)^n e^{-y\left(\eta+\mu^\nu\right)^n} y^{-\frac{\delta}{n}} \frac{n}{\delta} - \frac{\delta}{n} \frac{1}{\Gamma\left(1-\frac{\delta}{n}\right)} \int_0^\infty \mathrm{d}y \, \eta^n e^{-y\eta^n} y^{-\frac{\delta}{n}} \frac{n}{\delta} \\
&= \frac{1}{\Gamma\left(1-\frac{\delta}{n}\right)} \int_0^\infty \mathrm{d}z \, e^{-z} \left(\frac{z}{\left(\eta+\mu^\nu\right)^n} \right)^{-\frac{\delta}{n}} - \frac{1}{\Gamma\left(1-\frac{\delta}{n}\right)} \int_0^\infty \mathrm{d}z \, e^{-z} \left(\frac{z}{\eta^n} \right)^{-\frac{\delta}{n}} \\
&= \left(\eta+\mu^\nu\right)^\delta - \eta^\delta .
\end{aligned}
$$

From the above computation and Theorem 1.1, the claimed form for the Lévy measure easily follows. □

Remark 2.2. *Note that the determined measure can also be expressed by means of Riemann–Liouville fractional derivatives. Indeed, since [35, formula (2.117)]*

$$
-\frac{\delta}{n} \frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\frac{\delta}{n}\right)} = \frac{t^{-\left(1+\frac{\delta}{n}\right)}}{\Gamma\left(1-\left(1+\frac{\delta}{n}\right)\right)} = \left(D_{0+}^{1+\frac{\delta}{n}} 1\right)(t), \tag{2.11}
$$

where

$$
\left(D_{a+}^{\beta} f\right)(t) = \frac{1}{\Gamma(m-\beta)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_a^\infty \frac{f(y)}{(t-y)^{\beta-m+1}} \mathrm{d}y, \qquad \beta > 0,\ m = [\beta]+1, \tag{2.12}
$$

is the Riemann–Liouville fractional derivative [35], we obtain readily that

$$
m(\mathrm{d}x) = -\int_0^\infty e^{-\eta^n t} \mathbb{P}\left({}_\eta \mathcal{V}_t^{\nu,n} \in \mathrm{d}x \right) \left(D_{0+}^{1+\frac{\delta}{n}} 1 \right)(t) \, \mathrm{d}t. \tag{2.13}
$$

In order to study the operator associated to the subordinator ${}_\eta \mathfrak{V}_t^{\nu,\delta}$ we start from the Laplace transform

$$
\mathbb{E} e^{-\mu \, {}_\eta \mathfrak{V}_t^{\nu,\delta}} = e^{-t\left[(\eta+\mu^\nu)^\delta - \eta^\delta\right]}, \qquad t \geq 0,\ \mu > 0. \tag{2.14}
$$

By taking the derivative with respect to time we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E} e^{-\mu \, {}_\eta \mathfrak{V}_t^{\nu,\delta}} = -\left[(\eta+\mu^\nu)^\delta - \eta^\delta \right] \mathbb{E} e^{-\mu \, {}_\eta \mathfrak{V}_t^{\nu,\delta}}. \tag{2.15}
$$

From Theorem 2.1 we know that

$$
\left[(\eta+\mu^\nu)^\delta - \eta^\delta \right] = \int_0^\infty (1-e^{-\mu x}) m(\mathrm{d}x), \tag{2.16}
$$

where $m(\mathrm{d}x)$ is given by (2.9). Hence,

$$
\left[(\eta+\mu^\nu)^\delta - \eta^\delta \right] \mathbb{E} e^{-\mu \, {}_\eta \mathfrak{V}_t^{\nu,\delta}} = \int_0^\infty \left(\mathbb{E} e^{-\mu \, {}_\eta \mathfrak{V}_t^{\nu,\delta}} - e^{-\mu y} \mathbb{E} e^{-\mu \, {}_\eta \mathfrak{V}_t^{\nu,\delta}} \right) m(\mathrm{d}y), \tag{2.17}
$$

and equation (2.15) becomes

$$
\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E} e^{-\mu \, {}_\eta \mathfrak{V}_t^{\nu,\delta}} = -\int_0^\infty \left(\mathbb{E} e^{-\mu \, {}_\eta \mathfrak{V}_t^{\nu,\delta}} - e^{-\mu y} \mathbb{E} e^{-\mu \, {}_\eta \mathfrak{V}_t^{\nu,\delta}} \right) m(\mathrm{d}y). \tag{2.18}
$$

Now, a simple inversion of the Laplace transforms leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\big({}_\eta\mathfrak{V}_t^{\nu,\delta}\in\mathrm{d}x\big)/\mathrm{d}x=-\int_0^\infty\Big(\mathbb{P}\big({}_\eta\mathfrak{V}_t^{\nu,\delta}\in\mathrm{d}x\big)/\mathrm{d}x-\mathbb{P}\big({}_\eta\mathfrak{V}_t^{\nu,\delta}\in\mathrm{d}(x-y)\big)/\mathrm{d}(x-y)\Big)\,m(\mathrm{d}y). \tag{2.19}$$

Let us denote now $v(x,t)=\mathbb{P}\left({}_\eta\mathfrak{V}_t^{\nu,\delta}\in\mathrm{d}x\right)/\mathrm{d}x$ and with

$${}_\eta\Theta_x^{\nu,\delta}f(x,t)=\int_0^\infty\big(f(x,t)-f(x-y,t)\big)\,m(\mathrm{d}y) \tag{2.20}$$

the generating form of the operator. The probability density function $v(x,t)$ of ${}_\eta\mathfrak{V}_t^{\nu,\delta}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}v(x,t)=-{}_\eta\Theta_x^{\nu,\delta}v(x,t),\qquad x\ge 0,\ t\ge 0. \tag{2.21}$$

Remark 2.3. *We know [22, 17] that the Prabakhar derivative is defined, for a suitable set of functions [see 22, for details], as*

$$\mathbf{D}^\xi_{\alpha,\eta,\zeta;0+}f(t)=D_{0+}^{\eta+\theta}\int_0^t(t-y)^{\theta-1}E^{-\xi}_{\alpha,\theta}\left[\zeta(t-y)^\alpha\right]f(y)\,\mathrm{d}y, \tag{2.22}$$

with $\theta,\eta\in\mathbb{C}$, $\Re(\theta)>0$, $\Re(\eta)>0$, $\zeta\in\mathbb{C}$, $t\ge 0$. The fractional derivative appearing in (2.22) is the Riemann–Liouville fractional derivative with respect to time t and

$$E^\gamma_{\kappa,\varpi}(x)=\sum_{r=0}^\infty\frac{x^r(\gamma)_r}{r!\Gamma(\kappa r+\varpi)},\qquad \kappa,\varpi,\gamma\in\mathbb{C},\ \Re(\kappa)>0, \tag{2.23}$$

is the generalized Mittag–Leffler function (see e.g. [22]). The Laplace transform of the Prabakhar derivative is [15, 22]

$$\int_0^\infty e^{-st}\mathbf{D}^\xi_{\alpha,\eta,\zeta;0+}f(t)\,\mathrm{d}t=s^\eta(1-\zeta s^{-\alpha})^\xi\tilde f(s),\qquad s>0, \tag{2.24}$$

where $\tilde f$ is the Laplace transform of f. Therefore, with our choice of parameters,

$$\int_0^\infty e^{-st}\mathbf{D}^\delta_{\nu,\nu\delta,-\eta;0+}f(t)\,\mathrm{d}t=s^{\nu\delta}(1+\eta s^{-\nu})^\delta\tilde f(s)=(s^\nu+\eta)^\delta\tilde f(s),\qquad s>0. \tag{2.25}$$

This implies that ${}_\eta\Theta_x^{\nu,\delta}=\mathbf{D}^\delta_{\nu,\nu\delta,-\eta;0+}-\eta^\delta$ and thus

$$\int_0^\infty e^{-\mu t}{}_\eta\Theta_x^{\nu,\delta}v(x,t)\,\mathrm{d}x=\left[(\mu^\nu+\eta)^\delta-\eta^\delta\right]\tilde v(\mu,t). \tag{2.26}$$

Remark 2.4. *Note that an alternative representation for the operator ${}_\eta\Theta_x^{\nu,\delta}$ can be given in terms of Riemann–Liouville derivatives. We have*

$${}_\eta\Theta_x^{\nu,\delta}=(\eta+D^\nu_{0+})^\delta-\eta^\delta. \tag{2.27}$$

This can be proved by simply computing the Laplace transform, as follows:

$$\int_0^\infty e^{-\mu x}\left[(\eta+D^\nu_{0+})^\delta-\eta^\delta\right]f(x)\,\mathrm{d}x. \tag{2.28}$$

We can perform a formal expansion by means of Newton's theorem obtaining

$$\int_0^\infty e^{-\mu x}\sum_{r=1}^\infty\binom{\delta}{r}\eta^{\delta-r}D^{\nu r}_{0+}f(x)\,\mathrm{d}x=\left[(\mu^\nu+\eta)^\delta-\eta^\delta\right]\tilde f(\mu). \tag{2.29}$$

We considered the set of functions for which the semigroup property for the Riemann–Liouville derivative holds, and also that D^0_{0+} is the identity function. Note finally that (2.29) coincides with (2.26).

3 Generalization of the space-fractional Poisson process

In this section we present a study of possible generalizations of the so-called space-fractional Poisson process [33] (see also [34] for other more generalized results and special cases). Furthermore, in the following we will see how the process introduced in the previous section arises rather naturally in this framework as a time-change of an independent homogeneous Poisson process. Furthermore, Section 3.3 shows the effect of replacing the integer-order time derivative in the governing difference-differential equations with a non-local integro-differential operator with a three-parameter Mittag–Leffler function in the kernel, i.e. the so-called regularized Prabhakar derivative.

3.1 Space-fractional Poisson process

In order to make the paper as self-contained as possible, we summarize here the basic construction of the space-fractional Poisson process as it was carried out in [33]. The original aim was to generalize a homogeneous Poisson process in a fractional sense by introducing a fractional difference operator in the governing equations acting on the state space. The chosen operator is $(1-B)^\alpha$, $\alpha \in (0,1)$ (where B is the backward-shift operator, that is, given a function f_k depending on an index $k \in \mathbb{Z}$, $B(f_k) = f_{k-1}$ and $B^r(f_k) = B(B^{r-1}(f_k))$) which appears in the study of long memory time series. The introduction of the fractional difference operator $(1-B)^\alpha = \sum_{r=0}^\infty \binom{\alpha}{r}(-1)^r B^r$, implies a dependence of the probability of attaining a particular state to those of all the states below. The difference-differential equations for the state probabilities of the space-fractional Poisson process read

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} p_k(t) = -\lambda^\alpha (1-B)^\alpha p_k(t), & \alpha \in (0,1],\ \lambda > 0, \\ p_k(0) = \delta_{k,0}, \end{cases} \tag{3.1}$$

where $p_k(t) = \mathbb{P}\{N^\alpha(t) = k\}$, $k \geq 0$, $t \geq 0$, are the state probabilities of the space-fractional homogeneous Poisson process $N^\alpha(t)$, $t \geq 0$. If $\alpha = 1$, the process $N^\alpha(t)$ reduces to the homogeneous Poisson process of rate λ. The space-fractional Poisson process possesses independent and stationary increments and $\mathbb{E}(N^\alpha(t))^h = \infty$, $h = 1, 2, \dots$. The Cauchy problem (3.1) is easily solved using the probability generating function $G(u,t)$ and the Laplace transform. It turns out that the probability generating function can be written as

$$G(u,t) = e^{-\lambda^\alpha t (1-u)^\alpha}, \qquad |u| \leq 1, \tag{3.2}$$

and by means of a simple Taylor expansion the state probability distribution is recognized as a discrete stable distribution and reads

$$p_k(t) = \frac{(-1)^k}{k!} \sum_{r=0}^\infty \frac{(-\lambda^\alpha t)^r}{r!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)}, \qquad k \geq 0. \tag{3.3}$$

The behaviour of the process is made apparent by the subordination representation

$$N^\alpha(V_t^\gamma) \overset{\mathrm{d}}{=} N^{\alpha\gamma}(t), \qquad \alpha, \gamma \in (0,1). \tag{3.4}$$

For $\alpha = 1$, the time-changed process $N^1(V_t^\gamma)$ increases super-linearly with jumps of any integer size. Following the same lines, in the next section we will construct a generalized model by suitably adapting the operator acting on the state-space.

3.2 Generalized model

Consider the equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} p_k(t) = -\left\{ \left[\eta + \lambda^\nu (1-B)^\nu\right]^\delta - \eta^\delta \right\} p_k(t), & \nu n \in (0,1),\ n = \lceil \delta \rceil,\ \delta, \eta \in \mathbb{R}^+,\ \lambda > 0, \\ p_k(0) = \delta_{k,0}. \end{cases} \tag{3.5}$$

The generating function $G(u,t) = \sum_{k=0}^\infty u^k p_k(t)$ can be easily deduced from (3.5) by considering the following steps.

$$\frac{\partial}{\partial t} G(u,t) = -\sum_{r=1}^\infty \binom{\delta}{r} \eta^{\delta - r} \lambda^{\nu r} \sum_{m=0}^\infty \binom{\nu r}{m} (-1)^m \sum_{k=m}^\infty u^k p_{k-m}(t) \tag{3.6}$$

$$
\begin{aligned}
&= -\sum_{r=1}^{\infty} \eta^{\delta-r} \binom{\delta}{r} \lambda^{\nu r} \sum_{m=0}^{\infty} \binom{\nu r}{m} (-1)^m \sum_{k=0}^{\infty} u^{k+m} p_k(t) \\
&= -\sum_{r=1}^{\infty} \binom{\delta}{r} \eta^{\delta-r} \lambda^{\nu r} \sum_{m=0}^{\infty} \binom{\nu r}{m} (-u)^m G(u,t) \\
&= -G(u,t) \sum_{r=1}^{\infty} \binom{\delta}{r} \eta^{\delta-r} \lambda^{\nu r} (1-u)^{\nu r} \\
&= -G(u,t) \left[\left(\eta + \lambda^{\nu}(1-u)^{\nu}\right)^{\delta} - \eta^{\delta} \right], \qquad G(u,0) = 1.
\end{aligned}
$$

The solution is thus

$$
G(u,t) = e^{-t\left[\left(\eta+\lambda^{\nu}(1-u)^{\nu}\right)^{\delta} - \eta^{\delta}\right]}, \qquad |u| \le 1. \tag{3.7}
$$

The following theorem describes the structure of the point process for which the state probability distribution satisfies (3.5). For the proof we will make use of results described in Section 2.

Theorem 3.1. *Let ${}_{\eta}\mathfrak{V}_t^{\nu,\delta}$, $t \ge 0$, be the process in* (2.6) *and $N(t)$, $t \ge 0$, be a homogeneous Poisson process of rate $\lambda > 0$ independent of ${}_{\eta}\mathfrak{V}_t^{\nu,\delta}$. The probabilities $\mathbb{P}\{N({}_{\eta}\mathfrak{V}_t^{\nu,\delta}) = k\}$, $k \ge 0$, $t \ge 0$, of being in state k at time t, are solutions to* (3.5).

Proof. By exploiting the subordination property, the probability generating function of the process $N({}_{\eta}\mathfrak{V}_t^{\nu,\delta})$ can be written as

$$
\int_0^{\infty} \mathbb{E} u^{N(s)} \mathbb{P}\left({}_{\eta}\mathfrak{V}_t^{\nu,\delta} \in \mathrm{d}s\right). \tag{3.8}
$$

Then, by using Proposition 2.2, we obtain the probability generating function (3.7). ☐

The form of the probability generating function shows that the mean value of the associated process is infinite unless the subordinating process reduces to the tempered stable subordinator, i.e. if $\nu = 1$, $\delta \in (0,1)$. From the probability generating function $G(u,t)$, it is possible, by a simple Taylor expansion, to obtain the state probability distribution $p_k(t) = \mathbb{P}[N({}_{\eta}\mathfrak{V}_t^{\nu,\delta}) = k]$, $k \ge 0$, $t \ge 0$. We have the following theorem.

Theorem 3.2. *The state probability distribution $p_k(t) = \mathbb{P}[N({}_{\eta}\mathfrak{V}_t^{\nu,\delta}) = k]$, $k \ge 0$, $t \ge 0$, reads*

$$
\begin{aligned}
p_k(t) &= e^{t\eta^{\delta}} \frac{(-1)^k}{k!} \sum_{m=0}^{\infty} \frac{\Gamma(\nu m+1)}{m!\Gamma(\nu m-k+1)} \left(\frac{\lambda^{\nu}}{\eta}\right)^m \sum_{r=0}^{\infty} \frac{(-t\eta^{\delta})^r \Gamma(r\delta+1)}{r!\Gamma(r\delta-m+1)} \\
&= e^{t\eta^{\delta}} \frac{(-1)^k}{k!} \sum_{m=0}^{\infty} \frac{\Gamma(\nu m+1)}{m!\Gamma(\nu m-k+1)} \left(\frac{\lambda^{\nu}}{\eta}\right)^m {}_1\psi_1 \left[\left. \begin{matrix} (1,\delta) \\ (1-m,\delta) \end{matrix} \right| -\eta^{\delta} t \right],
\end{aligned} \tag{3.9}
$$

where ${}_h\psi_j(z)$ is the generalized Wright function (see [23], page 56, formula (1.11.14)).

Proof. Starting from the probability generating function $G(u,t)$, $t \ge 0$, $|u| \le 1$, we have

$$
\begin{aligned}
G(u,t) &= e^{t\eta^{\delta}} e^{-t[\eta+\lambda^{\nu}(1-u)^{\nu}]^{\delta}} = e^{t\eta^{\delta}} \sum_{r=0}^{\infty} \frac{[-t(\eta+\lambda^{\nu}(1-u)^{\nu})^{\delta}]^r}{r!} \qquad (3.10) \\
&= e^{t\eta^{\delta}} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \sum_{m=0}^{\infty} \binom{r\delta}{m} \eta^{\delta r-m} \lambda^{\nu m} (1-u)^{\nu m} \\
&= e^{t\eta^{\delta}} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \sum_{m=0}^{\infty} \binom{r\delta}{m} \eta^{\delta r-m} \lambda^{\nu m} \sum_{h=0}^{\infty} \binom{\nu m}{h} (-u)^h \\
&= \sum_{h=0}^{\infty} u^h e^{t\eta^{\delta}} \frac{(-1)^h}{h!} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \sum_{m=0}^{\infty} \frac{\Gamma(r\delta+1)}{m!\Gamma(r\delta-m+1)} \eta^{\delta r-m} \lambda^{\nu m} \frac{\Gamma(\nu m+1)}{\Gamma(\nu m-h+1)} \\
&= \sum_{h=0}^{\infty} u^h \left\{ e^{t\eta^{\delta}} \frac{(-1)^h}{h!} \sum_{m=0}^{\infty} \frac{\Gamma(\nu m+1)}{m!\Gamma(\nu m-h+1)} \lambda^{\nu m} \eta^{-m} \sum_{r=0}^{\infty} \frac{(-t)^r \eta^{\delta r} \Gamma(r\delta+1)}{r!\Gamma(r\delta-m+1)} \right\},
\end{aligned}
$$

and thus the claimed formula (3.9). ☐

Remark 3.1. *For $\delta = 1$ we obtain the discrete stable distribution (2.15) of [33] characterizing the behaviour of the space-fractional Poisson process. Indeed it suffices to compute*

$$\eta^{-m}e^{t\eta}\sum_{r=0}^{\infty}\frac{(-t\eta)^r\Gamma(r+1)}{r!\Gamma(r-m+1)} = \eta^{-m}e^{t\eta}\sum_{r=m}^{\infty}\frac{(-t\eta)^r}{(r-m)!} = \eta^{-m}e^{t\eta}(-t\eta)^m e^{-t\eta} = (-t)^m. \tag{3.11}$$

See also [34], Section 4.1. For general information on discrete stable random variables, the reader can refer to [39] or [13]. For $\nu = 1$, $\delta \in (0,1)$, a time-change given by a tempered ν-stable subordinator is retrieved (see [34], Section 4.2).

Remark 3.2. *Since $N(t)$ is a Lévy process, the subordinated process $N({}_\eta\mathfrak{V}_t^{\nu,\delta})$ is a Lévy process with Laplace exponent $\Phi(\mu) = (\eta+\lambda^\nu(1-e^{-\mu})^\nu)^\delta - \eta^\delta$, and thus it possesses stationary and independent increments. Moreover $N({}_\eta\mathfrak{V}_t^{\nu,\delta})$ is not in general a renewal process. This comes simply from [24] (see also [18]) and from the fact that, since ${}_\eta\mathfrak{V}_t^{\nu,\delta}$ is a properly rescaled and time-changed linear combination of independent stable subordinators, its inverse process $\mathfrak{E}_t = \inf\{w > 0 \colon {}_\eta\mathfrak{V}_w^{\nu,\delta} > t\}$ has in general non-stationary and dependent increments.*

Remark 3.3. *The reader can compare the state probabilities* (3.9) *with the results obtained in [34], Section 2, with a suitably specialized Bernstein function f.*

3.3 Time-fractional generalization with regularized Prabhakar derivatives

We saw from the above analysis (see Remark 2.3 and the subordination result) that the difference operator (3.5) is in practice connected with the Prabhakar derivative. This suggests that a further generalization, which can be still tractable, could involve a regularized Prabhakar derivative acting in time. This is what was considered in [17] in the case of the classical difference operator and in [15] for the generator of a Lévy process.

Let us first recall the definition of the regularized Prabhakar derivative.

Definition 3.1. *Let $\beta, \omega, \gamma, \alpha \in \mathbb{C}$, $\Re(\beta), \Re(\alpha) > 0$, $n = \lceil \Re(\beta) \rceil$, $f \in AC^n[0,b]$, $0 < b \le \infty$, where*

$$AC^n\,[a,b] = \left\{ f : [a,b] \to \mathbb{R} \colon \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} f(x) \text{ is absolutely continuous in } [a,b] \right\}.$$

The regularized Prabhakar derivative is defined as

$${}^C\mathbf{D}^\gamma_{\alpha,\beta,-\omega,0^+} f(x) = \mathbf{D}^\gamma_{\alpha,\beta,-\omega,0^+}\left[f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f^{(k)}(0^+) \right]. \tag{3.12}$$

For further details the reader can consult [17].

By means of the regularized Prabhakar derivative we can construct the following Cauchy problem.

$$\begin{cases} {}^C\mathbf{D}^\gamma_{\alpha,\beta,-\omega,0^+} p_k(t) = -\left\{ \left[\eta + \lambda^\nu(1-B)^\nu\right]^\delta - \eta^\delta \right\} p_k(t), & \nu n \in (0,1),\ n = \lceil \delta \rceil,\ \delta, \eta \in \mathbb{R}^+, \\ p_k(0) = \delta_{k,0}, \end{cases} \tag{3.13}$$

with the constraints $\omega > 0$, $\gamma \ge 0$, $0 < \alpha \le 1$, $0 < \beta \le 1$. We also have $0 < \beta\lceil\gamma\rceil/\gamma - r\alpha < 1$, $\forall\, r = 0, \dots, \lceil\gamma\rceil$, if $\gamma \ne 0$. With similar calculations as for (3.6), we obtain for $G(u,t) = \sum_k u^k p_k(t)$

$$\begin{cases} {}^C\mathbf{D}^\gamma_{\alpha,\beta,-\omega,0^+} G(u,t) = -\left\{ \left[\eta + \lambda^\nu(1-u)^\nu\right]^\delta - \eta^\delta \right\} G(u,t), & \nu n \in (0,1), \\ G(u,0) = 1. \end{cases} \tag{3.14}$$

By applying the Laplace transform with respect to time t we have that

$$\tilde{G}(u,s) = \frac{s^{\beta-1}(1+\omega s^{-\alpha})^\gamma}{s^\beta(1+\omega s^{-\alpha})^\gamma + (\eta + \lambda^\nu(1-u)^\nu)^\delta - \eta^\delta}, \tag{3.15}$$

which can be written, for $|[(\eta + \lambda^\nu(1-u)^\nu)^\delta - \eta^\delta]/[s^\beta(1+\omega s^{-\alpha})^\gamma]| < 1$,

$$\tilde{G}(u,s) = \sum_{n=0}^{\infty} \left\{ -[(\eta + \lambda^\nu(1-u)^\nu)^\delta - \eta^\delta] \right\}^n s^{-\beta n-1}(1+\omega s^{-\alpha})^{-n\gamma}. \tag{3.16}$$

We can now invert term by term the Laplace transform by using result (2.19) of [22] and Theorem 30.1 of [14]. The probability generating function can be written as

$$G(u,t)=\sum_{n=0}^{\infty}\left\{-[(\eta+\lambda^{\nu}(1-u)^{\nu})^{\delta}-\eta^{\delta}]t^{\beta}\right\}^{n}E^{\gamma n}_{\alpha,\beta n+1}(-\omega t^{\alpha}). \tag{3.17}$$

Note that formula (2.20) of [34] and equation (3.17) only coincide when $\gamma=0$ in (3.17) and, at the same time, in (2.20) of [34], $f(\cdot)$ is specialized to $(\eta+\lambda^{\nu}(\cdot)^{\nu})^{\delta}-\eta^{\delta}$.

By expanding the probability generating function (3.17) we can derive the state probability distribution for this generalized model. Indeed we have

$$G(u,t)=\sum_{n=0}^{\infty}(-t^{\beta})^{n}E^{\gamma n}_{\alpha,\beta n+1}(-\omega t^{\alpha})\sum_{r=0}^{n}(-\eta^{\delta})^{n-r}\sum_{m=0}^{\infty}\binom{r\delta}{m}\eta^{\delta r-m}\lambda^{\nu m}\sum_{h=0}^{\infty}\binom{\nu m}{h}(-u^{h}). \tag{3.18}$$

Rearranging, we get

$$p_k(t)=\frac{(-1)^k}{k!}\sum_{n=0}^{\infty}(-t^{\beta})^{n}E^{\gamma n}_{\alpha,\beta n+1}(-\omega t^{\alpha})\sum_{r=0}^{n}(-\eta^{\delta})^{n-r}\sum_{m=0}^{\infty}\binom{r\delta}{m}\eta^{\delta r-m}\lambda^{\nu m}\frac{\Gamma(\nu m+1)}{\Gamma(\nu m-k+1)}. \tag{3.19}$$

Remark 3.4. *Note that for $\gamma=0$, $\delta=1$, the generating function* (3.17) *reduces to that of the space-time fractional Poisson process [33] of parameters (β,ν), while for $\gamma=0$, $\nu=1$, $\delta\in(0,1)$, we obtain that of the tempered space-time fractional Poisson process. If we set $\gamma=0$, $\delta=1$, $\nu=1$, we get the time-fractional Poisson process of parameter β; for $\gamma=0$, $\delta=1$, $\beta=1$, we obtain the space-fractional Poisson process of parameter ν.*

We now show that the probabilities $p_k(t)$ of (3.19) are in fact state probabilities of a suitably time-changed homogeneous Poisson process. Consider the stochastic process, given as a sum of subordinated independent stable subordinators

$$\mathfrak{V}_t=\sum_{r=0}^{\lceil\gamma\rceil}V^{\beta\frac{\lceil\gamma\rceil}{\gamma}-r\alpha}_{\Phi(t)}, \qquad t\geq 0. \tag{3.20}$$

The random time change is defined as $\Phi(t)=\binom{\lceil\gamma\rceil}{r}V_t^{\frac{\gamma}{\lceil\gamma\rceil}}$, where $V_t^{\frac{\gamma}{\lceil\gamma\rceil}}$ is a stable subordinator independent of all the others and where $0<\beta\lceil\gamma\rceil/\gamma-r\alpha<1$ holds for each $r=0,1,\dots,\lceil\gamma\rceil$. The hitting time process can be defined in turn as

$$\mathfrak{U}_t=\inf\{s\geq 0\colon \mathfrak{V}_s>t\}, \qquad t\geq 0. \tag{3.21}$$

We are now ready to state the following theorem.

Theorem 3.3. *Let $\mathfrak{U}_t$, $t\geq 0$, be the hitting-time process in formula* (3.21). *Furthermore let $N({}_{\eta}\mathfrak{V}_t^{\nu,\delta})$ be a homogeneous Poisson process of parameter $\lambda>0$, subordinated by the process ${}_{\eta}\mathfrak{V}_t^{\nu,\delta}$, defined in* (2.6) *and independent of $\mathfrak{U}_t$. The time-changed process*

$$\mathfrak{N}(t)=N\left({}_{\eta}\mathfrak{V}^{\nu,\delta}_{\mathfrak{U}_t}\right), \qquad t\geq 0, \tag{3.22}$$

has state probabilities (3.19).

Proof. The claimed result can be proved by writing the probability generating function related to the time-changed process $\mathfrak{N}(t)$ as

$$\sum_{k=0}^{\infty}u^{k}\mathbb{P}(\mathfrak{N}(t)=k)=\int_0^{\infty}e^{-y[(\eta+\lambda^{\nu}(1-u)^{\nu})^{\delta}-\eta^{\delta}]}\mathbb{P}(\mathfrak{U}_t\in dy). \tag{3.23}$$

Therefore, by taking the Laplace transform with respect to time and taking into consideration Theorem 2.2 of [15] we have

$$\int_0^{\infty}\int_0^{\infty}e^{-y[(\eta+\lambda^{\nu}(1-u)^{\nu})^{\delta}-\eta^{\delta}]-st}\mathbb{P}(\mathfrak{U}_t\in dy)dt=\frac{s^{\beta-1}(1+\omega s^{-\alpha})^{\gamma}}{s^{\beta}(1+\omega s^{-\alpha})^{\gamma}+(\eta+\lambda^{\nu}(1-u)^{\nu})^{\delta}-\eta^{\delta}}, \tag{3.24}$$

which coincides with (3.15). □

As a byproduct of our analysis we obtain

$$\mathbb{E}e^{-\mu_\eta \mathfrak{V}^{\nu,\delta}_{\mathfrak{U}_t}} = \sum_{k=0}^{\infty} \left(- \left[(\eta + \lambda^\nu \mu^\nu)^\delta - \eta^\delta \right] t^\beta \right)^k E^{\gamma k}_{\alpha,\beta k+1}(-\omega t^\alpha), \qquad t \geq 0,\, s > 0, \tag{3.25}$$

that generalizes formula (3.37) of [3] with the Laplace exponent suitably specialized.

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